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Static and stationary 'cylindrically symmetric' Einstein–Maxwell fields, and the solutions of Van den Bergh and Wils

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Abstract. This paper considers stationary 'cylindrically symmetric' solutions of the Einstein–Maxwell equations whose metric takes 'block diagonal' form based on orbits of a two-parameter subgroup of the isometries and in which the Maxwell field lies in surfaces orthogonal to those orbits. It is shown that if the Maxwell field is non-null, it either inherits the metric symmetry or varies with one more of the coordinates in a specific way.

The cases in which the metric symmetry is inherited are discussed further. Their general solutions consist of three families in which the equations can be reduced to the equation for the third Painlevé transcendent followed by quadratures; one of these families is new, while the two such families given by Chitre *et al* are equivalent. It is shown how the particular solutions expressible in elementary functions (none of them new) arise. All previous solutions known to the author are identified. The calculations were done using the computer algebra system SHEEP.

The literature on this class of metrics is reviewed. In particular, the discussion given in the recent paper by Van den Bergh and Wils is amplified. The conditions for extra symmetry, and for the solutions to be static, are derived in a manner which clarifies their physical and mathematical origin, and relates the results to the methods for invariant classification of metrics developed in recent years.

1. Introduction

In their recent paper Van den Bergh and Wils (1983) (for brevity referred to hereafter as VW) used the method of Kinnersley (1977) to derive field equations for stationary cylindrically symmetric electrovacs (solutions of the Einstein–Maxwell equations), solved them for certain cases, and proved that in general the fields so obtained were not static. This stimulated me to re-examine the problem, and I present here some improvements of the VW results and those of other authors, together with a rediscussion of the earlier papers on the subject. In particular I prove that all solutions in the classes considered by VW and by Chitre *et al* (1975) (hereafter CGN) are either explicitly known or depend on the third Painlevé transcendental function.

In § 2 the general problem is stated, and the appropriate metrics written down: the full statement of the conditions required is unfortunately somewhat lengthy. The new result here is that in cylindrically symmetric stationary electrovacs with a metric of block diagonal form, non-null Maxwell fields which lie in the two-surfaces orthogonal to orthogonally transitive group orbits either inherit the metric symmetry or depend, in a particular way, on one extra variable.

The rest of the paper deals only with the case in which the symmetry of the metric is inherited by the Maxwell field. Section 3 contains the known particular solutions. In § 4 the field equations for the general case are discussed and simplified and in § 5 they are further reduced, in general to the form of the equation for the third Painlevé transcendent followed by quadratures. It is shown how the special cases for which solution in elementary functions is possible arise, and the relevant solutions in VW and elsewhere in the literature are identified. Section 6 rederives the VW results about the existence of additional Killing vectors and the conditions for the metric to be locally static, using methods of the kind discussed in Kramer *et al* (1980) (hereafter denoted KSMH and used as a standard reference to avoid too long a bibliography), Karlhede (1980) and Karlhede and MacCallum (1982). These methods are the theoretical basis of the recent work on the invariant classification of metrics, enabling different forms of the same metric, as presented in two coordinate systems, to be identified in a systematic manner (see MacCallum (1983) and references therein).

In the course of this paper a number of misprints, omissions and obscurities in relevant papers in the literature are touched on, and, where necessary, corrected.

The calculations reported in this paper (which could be checked by hand without impossible labour) were mostly carried out using the algebraic computing system SHEEP, written by I Frick of the University of Stockholm, and programs written for it by J Aman (also of Stockholm), by myself, and by G Joly (Queen Mary College, London).

2. The metrics to be considered

The term 'cylindrically symmetric stationary metric' means here a space–time which has three commuting Killing vectors that span a time-like three-dimensional surface. The metric is said to be static if there is a hypersurface-orthogonal time-like Killing vector. Krasinski (1978, unpublished) and Bonnor (1980) have discussed the existence of such a vector in the cylindrically symmetric stationary vacuum solutions of the Einstein equations (due to van Stockum (1937)); contrary to the statement of Som *et al* (1976), such a vector does not always exist in these spaces. Even if such a vector exists locally, the metric need not be globally static (Bonnor 1980), since the topological identification made in producing periodicity about an axis need not be compatible with the formation of the necessary hypersurfaces. The discussion in § 6 for the non-vacuum electrovac case, like that in VW, considers only the local stasis.

The assumption that there is actually an axis of symmetry (i.e. that the ignorable coordinate corresponding to one of the Killing vectors is periodic) is not essential to the local solution of the field equations, which is all this paper is concerned with, and the term 'cylindrically symmetric' is in that sense misleading. In particular the same metrics are sometimes described as plane symmetric, whether or not they have the additional rotational symmetry of the Euclidean plane, although most authors (e.g. KSMH) reserve the term for the case where this extra symmetry is present. Two commuting space-like symmetries can also be considered to act as a rotation and a boost (Bicak and Schmidt 1983).

The global properties are of importance when discussing the physical sources of the fields (e.g. as a current in a wire along an axis or a distribution of charge in a plane). It should also be noted, although such topics will not be fully discussed here, that conditions on the sources can be used to augment the conditions imposed on the metric. For example, Esposito and Glass (1976) state that static electrovac space-times can contain only electric, and not magnetic, fields (in the frame fixed by the hypersurface-orthogonal time-like Killing vector). However, this assumes in the definition of 'static' that currents are not allowed (because they break time reversal invariance) and nor are magnetic monopoles. This seems to me too restrictive, as it excludes the known classes of magnetostatic solutions. (Correspondingly, the theorem should not have appeared in KSMH, where it is quoted as theorem 16.4, since magnetostatic solutions *are* later discussed!)

The actual metrics considered in §§ 3–6 are subject to three further restrictions; none of these is necessary, in order to have an electrovac satisfying the above conditions, but they have been assumed in almost all previous work on the subject.

The first is that the metric has 'block diagonal' form, i.e. there are coordinates in which only two 2×2 submatrices of the matrix representing the metric contain non-zero entries, one of these being the metric of the orbits of a two-parameter Abelian group of motions; these orbits are said to have the property of orthogonal transitivity. For the case of stationary cylindrically symmetric metrics, this requirement could be formulated as the condition that one of the three Killing vectors is hypersurface orthogonal.

The existence of a 'block diagonal' form can be related to the nature of the sources. In KSMH, § 17.2, it is shown that if the four-current and the two Killing vectors of a stationary axisymmetric electrovac solution are coplanar then the metric must be of block diagonal form (note that the assumption used here, although sufficient, is not necessary); the proof uses a regularity condition at the axis of symmetry. In the present case, if a truly cylindrical solution were considered, the proof would exclude axial currents. I know of no proof that all cylindrically symmetric stationary electrovacs must take the block diagonal form, and the work of Harness (1982a, b) on general properties of metrics with a time-like hypersurface of homogeneity suggests that no such proof could be found.

The second is that the Maxwell field lies in the two-surfaces orthogonal to the group orbits with orthogonal transitivity.

These assumptions lead to two metrics. If the orbit of the two-dimensional group which has orthogonal transitivity is time-like, the metric takes the form assumed by VW,

$$ds^2=f(dt-w\,d\phi)^2-f^{-1}[e^{2\gamma}(dr^2+dz^2)+r^2\,d\phi^2]. \tag{2.1}$$

The corresponding metric for the case where the orthogonally transitive group acts on space-like surfaces, given by CGN, is

$$ds^2=f^{-1}[e^{2\gamma}(dt^2-dr^2)-r^2\,d\phi^2]-f(dz-w\,d\phi)^2. \tag{2.2}$$

The effect of the second assumption is to fix the form of the coefficient of $d\phi^2$ in (2.1) or (2.2) (see KSMH, §§ 15.1 and 20.1). All the metrics (2.2) are static.

The third extra assumption is that the electromagnetic vector potential lies in the orthogonally transitive group orbits and shares the symmetry of the metric. It is proved below that non-null Maxwell fields in the metric (2.1) [(2.2)] are either sinusoidal in z [respectively, t] or depend only on r; from this it follows (see KSMH, § 16.4) that the vector potential can be taken to lie in the orthogonally transitive orbits, and if the field is taken to be z [t] independent its vector potential (after a duality rotation if necessary) can be taken to depend only on r. Thus for non-null fields the third assumption essentially reduces to assuming that the field inherits the symmetry of the metric. (For discussions of the general question of such symmetry inheritance see e.g. KSMH, §§ 9.1 and 17.2, Catenacci *et al* (1982) and references therein.)

The vector potentials for (2.1) and (2.2) will therefore be taken, respectively, as

$$A_a\, dx^a = P\, dt + Q\, d\phi, \tag{2.3}$$

$$A_a\, dx^a = P\, dz + Q\, d\phi, \tag{2.4}$$

where P and Q depend on r alone. Equations (2.3) and (2.1) imply that the Maxwell field has in general a z component of magnetic field and an r component of electric field (taking $dt - w\, d\phi$ as the time axis), while (2.2) and (2.4) give Maxwell fields which correspond (with the time axis fixed by dt) to magnetic fields in the z, ϕ surfaces. The metrics will be vacuum if both P' and Q' are zero, where the prime denotes d/dr; in this paper only non-vacuum cases are considered.

I now have to prove the assertion that non-null Maxwell fields giving rise to (2.1) [or (2.2)] are either sinusoidal in z (t) or share the symmetry of the metric and that in the latter case they include one whose vector potential is of the form (2.3) [or (2.4)]; all other such solutions can be obtained from this one by a constant duality rotation. Maxwell fields in (2.1) which are dependent on z are discussed by Wils and Van den Bergh (1984). (In preparing this paper, I found that the example of this type given by Griffiths (1976), KSMH (11.63), is incorrect; subsequently, Dr Griffiths told me that this had already been noted by Repchenko (1978).)

The proof is based on the Rainich formulation (KSMH, § 5.4) which shows that a non-null Maxwell field is fixed by the metric up to a constant duality rotation. Duality rotations have the effect of exchanging contributions to the energy–momentum between electric and magnetic fields. Such a duality rotation applied to (2.3) [2.4] brings terms linear in z [t] into the vector potential. In particular, the overall sign of the Maxwell field is indeterminate in the solutions given below, corresponding to the possibility of a duality rotation through π, or the fact that gravity couples to a term quadratic rather than linear in the Maxwell field. I have omitted all the resulting sign ambiguities from subsequent formulae. The nature of the sources can be called on to restrict the choice of duality rotation by eliminating those parts of the field which (in a truly axisymmetric situation) would correspond to magnetic monopole sources.

The Rainich procedure is to first determine an 'extremal field' algebraically from the metric, and then find the 'complexion' α, the angular parameter of the duality rotation relating the actual field to the extremal field. The gradient of the complexion depends on the Ricci tensor and its derivatives and fixes α up to a constant.

The proof is given for (2.1), but applies equally, *mutatis mutandis*, to (2.2). The only non-zero off-diagonal component of the Ricci tensor of (2.1) in the orthonormal tetrad $(\sqrt{f}(dt - w\, d\phi),\ r\, d\phi/\sqrt{f},\ e^{\gamma}\, dr/\sqrt{f},\ e^{\gamma}\, dz/\sqrt{f}) = (\omega^1, \omega^2, \omega^3, \omega^4)$ is R_{12}. (The somewhat unorthodox numbering of axes is adopted for compatibility with Kinnersley's work: see § 4 below.) The form of (2.1) ensures $R^1{}_1 + R^2{}_2 = 0$, and the first Rainich condition, $R = 0$, then implies $R^3{}_3 + R^4{}_4 = 0$. A principal tetrad of the Ricci tensor can be found by a (r-dependent) Lorentz transformation ('boost') in the t, ϕ surfaces after which the extremal field would have F_{13} as its only independent non-zero component if $R_{33} < 0$, or F_{12} if $R_{33} > 0$.

The actual extremal field is related to this by the inverse of the r-dependent boost, and since all components of R_{ab} depend on r alone, so does the extremal field. Direct computation (with the help of SHEEP) shows that the gradient of the complexion has only a z component. Since $\alpha_{,r} = 0$, $\alpha_{,zr} = 0$, and hence $\alpha_{,z} = c$, which depends only on r, is constant, the complexion (up to a constant) must be of the form cz. The corresponding conclusion for (2.2) is that the complexion is ct, for constant c.

Hence either the Maxwell field inherits the metric symmetry ($c = 0$), or its components are sinusoidal in $z\,(t)$ and otherwise depend on r alone. Metrics of the second type are treated by Wils and Van den Bergh (1984). In the first case the external field, which can be taken to be the actual field, either has F_{13} and/or F_{23} non-zero and dependent on r alone, whence (2.3) is a suitable potential, or is related to such a field by a duality rotation through $\pi/2$.

It is obvious that the two classes (2.1)–(2.3) and (2.2)–(2.4) are related by complex transformations. In § 3 the special solutions expressible in terms of elementary functions are given: some of these form pairs related by complex transformations, but this is not always so since it may be impossible to change the constant parameters so that a real solution results. CGN showed that the general solutions of (2.2)–(2.4) depend on the third Painlevé transcendental function, in view of which it is not surprising that, as I show in §§ 4 and 5, the same is true of (2.1) and (2.2).

In fact I show that the two classes of such solutions given by CGN are the same (i.e. equivalent), but there are two distinct classes of solutions in the case (2.1), depending on the sign of a certain quantity (which vanishes exactly in McCrea's null solution), one of which was given by VW. Moreover, it is shown how the special solutions given in § 3 arise, and that they are the only possible ones.

3. The known particular solutions

In this section the known solutions for (2.1)–(2.4) expressible in terms of elementary functions are listed. It will be shown in § 5 that all such solutions are locally equivalent to one of the following (though distinctions between subclasses differing in global properties, e.g. periodicity of coordinates, might be of interest).

The solutions for the special case $w = 0$ (in (2.1) or (2.2)), which leads to diagonal static metrics, are, in the form (2.1) and (2.3),

$$f = G^{-2}, \qquad G = (kr^m + cr^{-m}), \qquad P = \frac{r}{pG}\frac{\mathrm{d}G}{\mathrm{d}r},$$
$$\mathrm{e}^{\gamma} = r^{m^2}, \qquad Q = 0, \tag{3.1}$$

where p, m^2 are real constants, and k and c are constants, which must obey $p^2 = -4kcm^2 > 0$ and be chosen so that G and P are real; or

$$f = G^{-2}, \qquad G = a\ln cr, \qquad P = -a/(\ln cr), \qquad \mathrm{e}^{\gamma} = 1, \qquad Q = 0, \tag{3.2}$$

where a and c are (real) constants; or

$$f = G^2 r^2, \qquad G = (kr^m + cr^{-m}), \qquad Q = \frac{r}{pG}\frac{\mathrm{d}G}{\mathrm{d}r},$$
$$\mathrm{e}^{\gamma} = r^{m^2}G^2 r, \qquad P = 0, \tag{3.3}$$

where p, m^2 are real constants, and k and c are constants, which must obey $p^2 = 4kcm^2 > 0$ and be chosen so that G and P are real.

The metric given by (3.3) can be interpreted as containing a magnetic field along the z direction, caused by current loops about the axis. However, the same metric and electromagnetic field, but with the names of the ϕ and z coordinates exchanged, can be interpreted as containing an azimuthal magnetic field (i.e. along the ϕ direction) due to a current along the axis! Equations (3.3) are related to (3.1) by a complex transformation, but no corresponding transformation of (3.2) leads to a real solution.

The solutions (3.1)–(3.3) can be considered to be special cases of the general form in which the function f and the electrostatic (or magnetostatic) potential in a static metric are functionally related, the functional relationship here following from the fact that all the functions depend on only one variable. (Of course, (2.1) with $w=0$ is not the general form of a static metric, so in this last remark f means just the length of the hypersurface-orthogonal time-like Killing vector.) The general case is discussed in KSMH, § 16.6.3. They can also, similarly, be considered to be members of Weyl's class of stationary axisymmetric electrovacs (Weyl 1917, KSMH § 19.1) and (3.2) is a member of the Papapetrou–Majumdar class (KSMH § 16.7).

Apart from these occurrences as special cases of general classes of solutions, the solutions (3.1)–(3.3) have been explicitly considered several times. The earliest paper known to me which gives them is Bonnor (1953), who found them in terms of electric rather than magnetic fields (i.e. duality-rotated through $\pi/2$) and used a slightly different form. They were also found from the Rainich formulation by Raychaudhuri (1960) (whose final form of (3.2) is incorrectly stated) and (3.3) was similarly found by Witten (1962), whose name is often attached to all these solutions. In KSMH, (3.3) is given as KSMH (20.9*a*) and (20.9*b*) and (3.1) is KSMH (20.9*c*). The solution (3.2) and the restrictions on parameters, as well as the forms of the electromagnetic field, are omitted from KSMH. A particularly well known special case is given by $m=1$ in (3.1), which after a coordinate transformation can be put in the standard form for the plane symmetric electrostatic solution, KSMH equation (13.26), due to McVittie, and includes Taub's plane symmetric vacuum solution. Further special cases are associated with the names of Bonnor, Kasner, Levi–Civita, Melvin, Mukherjee and others (see KSMH, §§ 13.4.4 and 20.2).

The other solutions known exactly and explicitly are the solution due to McCrea (1982) which has a null electromagnetic field, the special solution found by CGN and a new solution given by VW. (Note that the metric given by Wilson, KSMH (20.12), is not correct (McCrea 1982).)

McCrea's solution is given by (2.1) and (2.3) with

$$f=4q^2r^2+c_1r\ln kr, \qquad P=-qr, \qquad Q=0,$$
$$w=r/f, \qquad e^{2\gamma}/f=1/\sqrt{r}, \tag{3.4}$$

where q, c_1, and k are constants. This solution was independently rediscovered by Boachie and Islam (1983).

The CGN solution is given by (2.2) and (2.4) with

$$f=r^{2/3}, \qquad w=ar^{2/3}, \qquad e^{2\gamma}=r^{2/9}\exp(a^2r^{2/3}),$$
$$P=-ar^{2/3}/\sqrt{2}, \qquad Q=a^2r^{4/3}/\sqrt{8}, \tag{3.5}$$

where a is a constant.

The VW solution is given by (2.1) and (2.3) with

$$f=r^{2/3}, \qquad w=ar^{2/3}, \qquad e^{2\gamma}=r^{2/9}\exp(-a^2r^{2/3}),$$
$$P=-ar^{2/3}/\sqrt{2}, \qquad Q=a^2r^{4/3}/\sqrt{8}, \tag{3.6}$$

where a is a constant. This solution has also been found by R Jordan (University College, Dublin, unpublished) and by Islam (1983). (3.5) and (3.6) are related by a complex transformation.

4. The field equations

Following VW, I first use the Kinnersley (1977) form to derive equations governing the metrics (these are first integrals of the actual field equations, written in terms of potentials). Kinnersley writes the field equations in terms of the quantities f_{AB}, h_{MN}, ε_{AB}, ε_{MN}, ε^{AB}, ε^{MN} and the operators ∇ and $\tilde{\nabla}$, where the indices $(A, B\ldots)$ and $(M, N\ldots)$ can take the values 1, 2 and 3, 4 respectively, and, for the metric (2.1), considering quantities as vectors and matrices,

$$\varepsilon^{AB}=\varepsilon_{AB}=\varepsilon_{MN}=\varepsilon^{MN}=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},$$

$$f_{AB}=\begin{pmatrix}f & -fw\\ -fw & fw^2-r^2/f\end{pmatrix}, \qquad h_{MN}=f^{-1}\,\mathrm{e}^{2\gamma}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \tag{4.1}$$

$$\nabla=\frac{\partial}{\partial x^M}=\left(\frac{\partial}{\partial r},\frac{\partial}{\partial z}\right), \qquad \tilde{\nabla}=\delta_{NP}\varepsilon^{PQ}\nabla=\left(\frac{\partial}{\partial z},-\frac{\partial}{\partial r}\right),$$

where $\delta_{MN}=\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$, which is obviously conformally related to h_{MN}, and indices are raised and lowered with the ε using the sign convention fixed by

$$f^A{}_B=\varepsilon^{AC}f_{CB}, \qquad f_{CB}=\varepsilon_{EC}f^E{}_B,$$

(the sign conventions for ε and raising and lowering are not explicitly stated by Kinnersley: I have used the choice in KSMH, which is consistent with Kinnersley's later equations). The potential (2.3) can be written as $A_C\,\mathrm{d}x^C$, where $A_C=(P,Q)$. The matrix $r^{-1}f_C{}^D$ is its own inverse, as is easily verified by direct computation.

Kinnersley shows that the field equations can be written in terms of potentials B_E, ψ_{FG} (taking units so that the relevant coupling constants are simple), as

$$\nabla A_C=r^{-1}f_C{}^D\tilde{\nabla}B_D, \tag{4.2}$$

$$\nabla f_{CD}=r^{-1}f_C{}^E(\tilde{\nabla}\psi_{ED}+2A_E\tilde{\nabla}B_D+2A_D\tilde{\nabla}B_E), \tag{4.3}$$

together with equations from which γ can be obtained by quadrature.

Putting in the assumption that the metric and Maxwell potential depend only on r, VW show that the $M=4$ components of these equations imply that the potentials B and ψ are independent of r and the $M=3$ equations show that their dependence on z is linear with constant coefficients. They then take the specific components which yield first-order differential equations for Q and w, write out certain of the original Einstein–Maxwell equations, and, integrating one of them, obtain a first-order equation for P and a second-order equation for f. However, (4.2) and (4.3) immediately give four first-order equations for f, w, P and Q, together with two consistency conditions: these appear explicitly in Kinnersley (1977) as (components of) his equations (7.1)–(7.10). Writing

$$B_E=(q,p)z, \qquad \psi_{CD}=\begin{pmatrix}c_1 & c_2\\ c_3 & c_4\end{pmatrix}z,$$

(where the c_i and p and q are constants) these equations, for the metric (2.1) with (2.3), are

$$P' = -f(p+qw)/r, \tag{4.4}$$

$$Q' = -qr/f - wP', \tag{4.5}$$

$$f' = -f[w(c_1+4qP)+c_3+2pP+2qQ]/r, \tag{4.6}$$

$$w' = r(c_1+4qP)/f^2, \tag{4.7}$$

$$c_2 = c_3+2,$$

$$c_4 = (c_1+4qP)(r^2/f^2-w^2)-4pQ-2w(1+c_3+2pP+2qQ),$$

and the equation for γ is

$$\gamma' = (Q'+wP')^2 f/r - rP'^2/f - w'^2 f^2/4r + rf'^2/4f^2. \tag{4.8}$$

This system of equations is still somewhat daunting; to simplify it further I use (in § 5) the gauge transformations of the potentials and the remaining freedom of choice of the coordinates.

The equations for (2.2) can be found from a treatment corresponding to that just given for (2.1). (CGN actually worked by integrating the equations arising from the canonical Hamiltonian formalism, specialised to the current problem.) The equations are

$$P' = -f(p+qw)/r, \tag{4.9}$$

$$Q' = qr/f - wP', \tag{4.10}$$

$$f' = f[w(c_1+4qP)+c_3+2pP+2qQ]/r, \tag{4.11}$$

$$w' = r(c_1+4qP)/f^2, \tag{4.12}$$

$$c_2 = c_3+2,$$

$$c_4 = -(c_1+4qP)(w^2+r^2/f^2)-4pQ-2w(-1+c_3+2pP+2qQ),$$

where to recover the actual form in CGN one must first alter units so that the coupling constant of electromagnetism to gravitation is divided by 4 and then make the replacements $P \to A_z$, $Q \to A_\phi$, $p \to -c$, $q \to c_2$, $w \to -\sigma$, $c_1 \to -c_3$, $c_3 \to c_4$ (confusingly!) The equation corresponding to (4.8) is

$$\gamma' = (Q'+wP')^2 f/r + rP'^2/f + w'^2 f^2/4r + rf'^2/4f^2. \tag{4.13}$$

In both cases the solution is a vacuum solution if $p = 0 = q$. From now on I shall assume that not both of p and q vanish.

5. Solving the field equations

When obtaining the solutions of the equations just given, it is very useful to eliminate as far as possible the coordinate and gauge freedoms which can lead to the presentation of apparently different forms of the same metrics. It turns out to be sufficient to consider only a few simple transformations. I take (2.2) first, since most of the steps in this case were already worked out by CGN.

Let us take the case where q is non-zero first The transformations are presented with the new quantities marked by an overbar, which is dropped before the next step, and quantities which are unaltered are not mentioned. A transformation

$$\bar{z} = z + p\phi/q, \qquad \bar{w} = w + p/q, \qquad \bar{Q} = Q - pP/q, \tag{5.1}$$

leaves the equations in the same form but with $p = 0$. Moreover we are free to make a (trivial) gauge transformation

$$\bar{P} = P - c_1/4q \tag{5.2}$$

to set $c_1 = 0$, followed by a similar change of Q to set $c_3 = 0$. Finally, we can use a transformation

$$\bar{\phi} = k\phi, \qquad \bar{Q} = Q/k, \qquad \bar{w} = w/k, \qquad \bar{r} = r/k, \qquad \bar{q} = kq, \tag{5.3}$$

to set q to any desired (non-zero) value.

CGN treat the case $q = 0$ as a second distinct family of solutions. However, by the transformation

$$\begin{aligned} &\bar{z} = \phi, \qquad \bar{\phi} = z, \qquad \bar{f} = fw^2 + r^2/f, \qquad \bar{w} = fw/\bar{f}, \\ &\qquad \bar{Q} = P, \qquad \bar{P} = Q, \qquad \bar{p} = q = 0, \qquad \bar{q} = p, \\ &\qquad \bar{c}_1 = c_4, \qquad \bar{c}_4 = c_1, \qquad \bar{c}_3 = -c_2, \qquad \bar{c}_3 = -c_2, \end{aligned} \tag{5.4}$$

(which is merely renaming coordinates and has no physical effect), followed, as necessary, by (5.2) and (5.3), one can map the first case into the second. The two families given by CGN are thus locally equivalent. (For the correspondence with their paper one must replace the notation used here by using the correspondence given in § 4 together with $1/\bar{f} \to y$, $\bar{w} \to f$, $f \to x$.) Note that the transformations (5.1) and (5.3) would not preserve the required global properties for z and ϕ in a solution that was actually cylindrically symmetric, so that at the end the solution would have to be transformed by the inverse of whatever series of transformations of the form (5.1)–(5.4) had been applied.

Taking the equations (4.9)–(4.12) in the form in which $q = 0$, we have

$$P' = -fp/r, \tag{5.5}$$

$$Q' = -wP', \tag{5.6}$$

$$f' = f(wc_1 + c_3 + 2pP)/r, \tag{5.7}$$

$$w' = rc_1/f^2, \tag{5.8}$$

where a gauge transformation of P could be used to set $c_3 = 0$. (Note this cannot be done in the vacuum case.) Differentiating (5.7) and substituting from (5.5) and (5.8) gives

$$(rf'/f)' = (c_1)^2 r/f^2 - 2p^2 f/r, \tag{5.9}$$

which on making the substitution $r \to \bar{r}^2$, $f \to \bar{r}/y$, gives immediately the third of the six Painlevé equations which define new transcendental functions, in the canonical form given by Ince (1926, p 345). In principle, once a solution of (5.9) is found, one can obtain P, w, Q and γ by successively integrating (5.5), (5.8), (5.6) and (4.13). However, this is probably of little practical value (though one could perhaps investigate physical properties at an axis using asymptotic properties of the third Painlevé transcendent functions). Note also that since r is (up to a constant multiple) invariantly defined, one cannot remove all transcendental functions simply by a coordinate transformation.

(5.9) does have non-transcendental solutions for special choices of the constants. There are three cases. One is $p=0$, which is the vacuum solution which is locally static (Bonnor 1980, case I). The second is $c_1=0$, which gives $w=0$, and yields (3.3); the similarity between (3.3) and the form of the vacuum solution, KSMH (20.7), is not so surprising in view of the way they rise from (5.9). The third is the case where the right-hand side of (5.9) vanishes, which (using a transformation of scale of z to absorb a constant) yields the form (3.5), where $c_1 = 2a/3 = \sqrt{2}p$. Hence these are the only electrovac or vacuum solutions of the form (2.2), (2.4) which are integrable in terms of elementary functions.

We now apply similar methods to the VW case. The essential difference stems from the change of signature. In the case $q=0$, having used a gauge change of P to set $c_3=0$, one easily gets again an equation of the form (5.9) but with the overall sign of the right-hand side reversed. This is the class found by VW who also gave the special case (3.6) which arises in the same way as (3.5). The corresponding static solutions are (3.1) and (3.2). The class with q non-zero shows more variety. One can first use

$$\bar{t} = t + k\phi, \qquad \bar{p} = p + kq, \qquad \bar{w} = w + k, \qquad \bar{Q} = Q - kP, \tag{5.10}$$

to set $p=0$. The further step depends on the result of exchanging the variables ϕ and t. If the quantity

$$Y = r^2/f - fw^2 \tag{5.11}$$

is negative, the ϕ we are using is a time variable, and we just find the same family of solutions again. If the Y of (5.11) is zero, the equations can be integrated to obtain McCrea's solution (3.4). If the Y of (5.11) is positive, the exchange of t and ϕ exchanges time and space directions. The equation corresponding to (5.9) is

$$(rY'/Y)' = -(c_4)^2 r/Y^2 - 2q^2 Y/r, \tag{5.12}$$

where the metric has been taken as

$$ds^2 = -Y(d\phi - x\, dt)^2 - Y^{-1}[e^{2\gamma}(dr^2 + dz^2) - r^2\, dt^2]. \tag{5.13}$$

Note that the static case arising is now (3.3) and no analogue of the special solutions (3.5) and (3.6) is possible.

I should say that I would perhaps not have determined the equivalence and non-equivalence of the various classes correctly without the aid of SHEEP.

The classes arising from (2.1) and (2.3) are the electrovac generalisations of the vacuum solutions which are not static (case III in Bonnor (1980)) while the class arising from (2.2) and (2.4) is the electrovac generalisation of the (locally) static class of vacuum solutions (case I in Bonnor (1980)). The generalisation of the third type of van Stockum solution (Bonnor (1980), case II and KSMH (18.23)) is (3.4), as McCrea himself noted, and is (cf Boachie and Islam 1983) a special case of the general electrovac solution with a null Killing vector found by Kramer (KSMH (21.33)–(21.35)), the null Killing vector being that corresponding to the ignorable coordinate ϕ as chosen above (up to the usual sign ambiguity).

I now relate these results to those of VW. VW treat first the case $q = c_1 = 0$. They duly obtain the static solutions, but do not identify them with (3.1) and (3.2), with which they are identical. (VW also speak of a third constant of integration, but this can be absorbed by a change of gauge of P.) The remark they make concerning the idea of superposition of the static solutions is related to some work by Safko (1977). Unfortunately, Safko, at the critical poiint of his calculation, made a substitution of the form (his equation (3.1))

$$d\bar{z} = \sin\varepsilon\, r\, dH + \cos\varepsilon\, dz \tag{5.14}$$

and considered this as a coordinate transformation. However, the right-hand side does not have a zero exterior derivative, and hence this form is not integrable for $\bar{z}$ as Safko supposes. The explicit metric he gives is in fact not an electrovac, except, of course, in the special case $\varepsilon = 0$, which is the original static metric he used. (I have directly verified this with SHEEP.)

The second case in VW is $q = 0$. They reduce this case to the Painlevé equation, and give the particular solution (3.5) (in a slightly more complicated form from which (3.5) can be derived by rescaling t and applying a transformation of the type (5.10)).

Finally, VW treat the case where q is non-zero, but could only find the special solutions. The first case they give is where $(Q - kP)' = 0$. This implies that after the transformation (5.10) using the given constant k

$$qr/f = w(p + qw)f/r. \tag{5.15}$$

If the Y of (5.11) is not now zero (which implies $p = 0$ and gives McCrea's solution (3.4)), then, on rewriting the metric as (5.13), where Y could take either sign, we find

$$x = fw/(r^2/f - fw^2) = q/p$$

is a constant, which can immediately be transformed away. The result is the solution (3.3). This shows that the solutions of Arbex and Som (1973) quoted by VW are in fact the only solutions of this type (except for McCrea's solution).

VW also explore the possibility that there are special cases in which (removing a pure gauge term from their form) P is some power of r. This leads back to the solution (3.6) but in an apparently different form which VW did not identify.

Finally, how can one be sure that the classes mentioned actually are distinct? The first point is that the r direction is invariantly defined. One can thus distinguish those cases in which the time-like eigenplane of the Ricci tensor includes the r direction and those where it does not, and this separates the two distinct cases of (2.1). In the duality choice implied by (2.3) one can express the difference in terms of whether, after the transformations described, the field reduces to a pure electric field in the r direction or a pure magnetic field in the z direction. Secondly, the tetrad used after the transformations is one in which the Ricci tensor is diagonal, and the Petrov type I Weyl tensor takes a canonical form; it is thus completely and invariantly determined (up to reflections). Hence one can use the relations between the components of the Weyl tensor to distinguish between the CGN and VW classes.

6. Additional symmetry and invariant properties

VW consider the possibility of extra Killing vectors by integrating the Killing equations. One can however obtain the result immediately by inspecting the invariants of the spaces following Karlhede (1980) and Karlhede and MacCallum (1982). As before, I consider only cases in which one of p or q is non-zero (i.e. I ignore the vacuum fields which were covered by Bonnor (1980)). The Maxwell tensor invariant for the non-null cases is not identically zero, and hence nor are the eigenvalues of the Ricci tensor. In fact they take values of the form (where k stands for p or q)

$$e^{-2\gamma}f^2r^{-2}k^2, \tag{6.1}$$

in the canonical tetrad chosen as described earlier, which, since $fe^{-\gamma}$ is in general a transcendental function cannot be constant (one could check the form of its derivative). For the special cases where exact integrals are available, one can again easily check that (6.1) is not constant. This immediately implies that there are no extra translational Killing vectors.

For those cases with a non-null electromagnetic field, the only possible isotropies would be composed of a boost in the time-like eigenblade and a spatial rotation in the space-like eigenblade (KSMH, chap 5 and 9). However, these are in general both ruled out by the fact that the metrics are of Petrov type I (with a value of the invariant N used in Petrov classification which fills nearly two pages of computer output but can easily be seen on inspection to vanish only if there is no Maxwell field or an algebraic relation exists between the transcendental functions). The special cases are also of Petrov type I, except for the plane symmetric case included in (3.1) which is of Petrov type D, and the McCrea solution which, like the more general class of which it is a member, is of Petrov type II and again can have no extra Killing vectors (as McCrea pointed out).

If in the case (2.1) there were a static Killing vector it would have to be a Ricci and Weyl eigendirection (KSMH, § 16.6.1); as stated in § 5, the time axis of the tetrad chosen above is the unique direction with these properties, and hence any static Killing vector must lie along that time axis. Hence the metric is static only if $w=0$ in the implied choice of coordinates, which immediately reduces to the static cases already mentioned (a result in full agreement with VW). A similar argument rules out the existence of any extra hypersurface-orthogonal Killing vector in the CGN case.

7. Concluding remarks

In § 5 it was shown that all electrovacs of a stationary cylindrically symmetric character (in the sense discussed in § 1) which have metrics and Maxwell fields given by (2.1) and (2.3) or (2.2) and (2.4) are either derived from a Painlevé transcendent function of the third kind, or are (locally) equivalent to one of the solutions given by (3.1)–(3.6): the solutions which are of the form involving transcendental functions fall into three distinct classes. Since the possibilities arising from (2.1)–(2.4) are thus exhausted, it has been possible to identify all solutions in the literature known to me which are given in terms of elementary functions explicitly with one or other of (3.1)–(3.6).

That is not to say the interest of the solutions is exhausted. In § 6 I dealt with the possibility of an extra Killing vector and with the possibility of static solutions, showing that the solutions for the case (2.1) are non-static (ie. do not admit a locally hypersurface-orthogonal time-like Killing vector) unless they are locally equivalent to the metrics given by (3.1) or (3.2), and that in no case do the non-vacuum fields admit any extra Killing vectors, except for the plane symmetric subcase of (3.3) which is of Petrov type D.

However, the physical understanding and the global restrictions arising from the assumption that one of the coordinates is periodic remain to be investigated. Moreover, very powerful methods exist for generating electrovac solutions with two commuting Killing vectors from any given such solution, and no attempt is made here to explore or exploit these beyond the simple transformations used in § 5. The methods cannot lead to anything new within the classes studied in this paper, since all their solutions are covered here, but could relate the solutions studied to each other or to interesting solutions in other classes. For some introduction to these methods the reader is referred to KSMH, chap 30, Kinnersley (1977) and Cosgrove (1980, 1982), which contain numerous references to the extensive literature.

It is hoped to return to the matters of the global restrictions, physical interpretation and use of generating techniques at some future time. I cannot, however, resist the speculation that one may be able to transform these families of solutions into other known families where the third Painlevé transcendent arises (e.g. those given by Maartens and Nel (1978)).

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